Cantor sets and dimension
Problem set 1

1. Let $T : X \mapsto Y$ be a surjective α-Hölder map of metric spaces, i.e. for some constant $C > 0$, $d_Y(T(x_1), T(x_2)) \leq C d_X(x_1, x_2)^\alpha$. Show that
$$\operatorname{Hdim} Y \leq \frac{\operatorname{Hdim} X}{\alpha}.$$

2. Use the previous problem and the Cantor staircase map of the standard Cantor set to the unit interval to obtain an alternative derivation of the Hausdorff dimension of the Cantor set.

3. Show that the definition of the Minkowski and Hausdorff dimensions for subsets of $\mathbb{R}^d$ do not change if one considers only coverings by b-adic cubes.

4. Let $l_0 = 1$, $0 < l_n < l_{n-1}/2$ be a sequence of positive numbers. Define a Cantor set $C = \bigcap_{n \geq 0} C_n$ in $\mathbb{R}^d$ by taking C_0 to be the unit cube, and obtaining C_n from C_{n-1} by selecting 2^d corner cubes of size l_n from each cube comprising C_{n-1}. What is the Hausdorff dimension of C? More generally, for which gauge functions h we have $H_h(C) > 0$? $m_h(C) < \infty$?

5. Let $K \subset \mathbb{R}^d$ be a bounded set, $\varepsilon > 0$, and $K^\varepsilon := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq \varepsilon\}$. Show that
$$\limsup_{\varepsilon \to 0} \frac{\log \operatorname{Volume}(K^\varepsilon)}{|\log \varepsilon|} = d + \overline{\operatorname{Mdim}}\, K,$$
$$\liminf_{\varepsilon \to 0} \frac{\log \operatorname{Volume}(K^\varepsilon)}{|\log \varepsilon|} = d + \underline{\operatorname{Mdim}}\, K,$$

6. For a set K in a metric space define *α-packing premeasure of K* as
$$\hat{\mathcal{P}}^\alpha(K) = \lim_{\varepsilon \to 0} \left(\sup \sum_{j=1}^{\infty} r_j^\alpha \right),$$
where the supremum is taking over all collections of disjoint balls $\{B(x_j, r_j)\}$ with $x_j \in K$ and $r_j < \varepsilon$.

   (a) Show that if $\hat{\mathcal{P}}^\alpha(K) < \infty$ and $\beta > \alpha$ then $\hat{\mathcal{P}}^\beta(K) = 0$.

   (b) Show that if $\hat{\mathcal{P}}^\alpha(K) < \infty$ then $\overline{\operatorname{Mdim}}(K) \leq \alpha$.

   (c) Show that for any M, we have
$$\sum_{m \geq M} P(2^{-m}, K) 2^{-m\alpha} \geq 4^{-\alpha-1} \hat{\mathcal{P}}^\alpha(K).$$

   Hint: Consider the packing by disjoint balls with maximal radius 2^{-M-2} and
$$\sum_{j=1}^{\infty} r_j^\alpha > \hat{\mathcal{P}}^\alpha(K)/4.$$
   Notice that this cover cannot contain more then $P(2^{-m}, K)$ balls of radius between 2^{-m-1} and 2^{-m}.

(d) Conclude that if $\hat{\mathcal{P}}^\alpha(K) > 0$ then $\overline{\mathrm{Mdim}}(K) \geq \alpha$ and thus

$$\overline{\mathrm{Mdim}}(K) = \inf\{\alpha : \hat{\mathcal{P}}^\alpha(K) = 0\} = \sup\{\alpha : \hat{\mathcal{P}}^\alpha(K) = \infty\}$$

7. For a set K in a metric space define *α-packing measure of* K as

$$\mathcal{P}^\alpha(K) = \inf \left\{ \sum_{j=1}^{\infty} \hat{\mathcal{P}}^\alpha(K_j) \,:\, K \subset \bigcup_{j=1}^{\infty} K_j \right\}.$$

(a) Show that $\mathcal{P}^\alpha$ is a metric outer measure, and thus define a regular measure on Borel sets.

(b) Define the *packing dimension* of a set K by

$$\mathrm{Pdim}(K) = \inf\{\alpha : \mathcal{P}^\alpha(K) = 0\}.$$

Show that

$$\mathrm{Pdim}(K) = \inf \left\{ \sup_j \overline{\mathrm{Mdim}}(K_j) \,:\, K \subset \bigcup_{j=1}^{\infty} K_j \right\}.$$

(c) Show that it is enough to consider covers by closed sets.

(d) Show that $\mathrm{Pdim}(K) \geq \mathrm{Hdim}(K)$.

(e) Construct a set K with $\mathrm{Pdim}(K) < \underline{\mathrm{Mdim}}(K)$.